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ON THE EXCITATION OF A PRESTRESSED CYLINDER*

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The linearized theory of propagation of elastic waves /1,2/ is used to develop a method of investigating the specific features of an infinite, prestressed circular cylinder excited by a rigid strap vibrating at its surface. The material of the cylinder is assumed to be compressible, initially isotropic, with elastic potential of an arbitrary form. The axisymmetric oscillations of the strap are harmonic, of frequency ω. The study is based on the method of reducing a boundary value problem of the theory of elasticity to an integral equation, or to a system of integral equations, depending on the conditions of the contact between the strap and the cylinder.

Numerical analysis of the properties of the integral operator symbols makes it possible to study the influence of the initial deformation intensity, as well as of the choice of the elastic potential form, on the basic characteristics of the wave process taking place within the cylinder. A more detailed analysis can be made in the course of constructing solutions of the integral equations. In particular, the problem of a radially oscillating strap can be used to show the effect of the magnitude of the initial stresses on the distribution of contact stresses under the stamp, and on the behavior of the free surface outside it.

1. The initial state of stress is assumed homogeneous, i.e. /1,2/

$$u_n^\circ = \delta_{in}(\lambda_i - 1)x_n, \quad n = 1, 2, 3$$
$$\lambda_i = \text{const}, \lambda_1 = \lambda_2 \neq \lambda_3, \sigma_{11}^{*\circ} = \sigma_{22}^{*\circ} \neq \sigma_{33}^{*\circ}$$

Here u_n° denote the components of the initial displacement vector, $\sigma_{ii}^{*\circ}$ are the components of the generalized initial stress tensor, λ_i denote the relative elongations of the fibers and δ_{in} is the Kronecker delta.

Let us pass to the cylindrical r, φ, z-coordinate system, directing the Oz-axis along the Ox_3-axis. Using the principle of limit absorption /3,4/ we can reduce the problem of axisymmetric excitation of a cylinder by a virbrating strap to the study of the equations ($u_r(z, t) = U_r(z)e^{-i\omega t}$, $w(z, t) = W(z)e^{-i\omega t}$ are respectively the radial and axial component of the displacement vector, $q(z)$ and $\tau(z)$ are the normal and tangential components of the contact stress vector, $R = 1$ is the cylinder radius and a denotes the half-width of the strap)

$$U_r(z) = \frac{1}{2\pi\mu}\int_{-a}^{a} k_{11}(z-\xi)q(\xi)\,d\xi + \frac{i}{2\pi\mu}\int_{-a}^{a} k_{12}(z-\xi)\tau(\xi)\,d\xi \tag{1.1}$$

$$W(z) = \frac{i}{2\pi\mu}\int_{-a}^{a} k_{21}(z-\xi)\tau(\xi)\,d\xi + \frac{1}{2\pi\mu}\int_{-a}^{a} k_{22}(z-\xi)q(\xi)\,d\xi$$

$$k_{jn}(t) = \int_\Gamma K_{jn}(u)e^{iut}\,du \tag{1.2}$$

$$K_{jn}(u) = \Delta_{jn}(u, \varkappa_2)\cdot\Delta^{-1}(u, \varkappa_2), \quad j, n = 1, 2 \tag{1.3}$$
$$\Delta_{11}(u, \varkappa_2) = A_1A_7A_5^{-1}\sigma_1\sigma_2(\sigma_2^2 - \sigma_1^2)I_{11}I_{12}$$
$$\Delta_{12}(u, \varkappa_2) = \sigma_2 l_1 I_{01}I_{12} - \sigma_1 l_2 I_{02}I_{11}$$

$$\Delta_{21}(u, \varkappa_2) = d_1\sigma_2 m_2 I_{01}I_{12} - d_2\sigma_1 m_1 I_{11}I_{02} \tag{1.4}$$
$$\Delta_{22}(u, \varkappa_2) = (d_2l_1 - d_1l_2)I_{01}I_{02} + f\sigma_1 d_2 I_{11}I_{02} - f\sigma_2 d_1 I_{01}I_{12}$$
$$\Delta(u, \varkappa_2) = l_1m_2\sigma_2I_{01}I_{12} - l_2m_1\sigma_1I_{02}I_{11} + fA_1A_7A_5^{-1}\sigma_1\sigma_2(\sigma_2^2 - \sigma_1^2)I_{11}I_{12}$$

$$l_i = A_1\sigma_i^2 - \lambda_1\lambda_3 a_{13}d_i \quad f = \lambda_1^2 a_{13} - A_1 \tag{1.5}$$
$$m_i = \lambda_1\lambda_3\mu_{13}u^2 + A_7d_i \quad d_i = (A_1\sigma_i^2 + S_1)A_5^{-1}$$
$$S_1 = \mu_{33}\varkappa_2^2 - A_3u^2 \quad S_2 = \mu_{33}\varkappa_2^2 - A_6u^2$$

$$\sigma_{1,2}^2 = 0{,}5\,(D_2 \mp \sqrt{\Sigma}) \quad \Sigma = D_2^2 - 4D_1D_3$$
$$D_1 = A_1A_7, \quad D_2 = A_5u^2 - A_1S_1 - A_7S_2, \quad D_3 = S_1S_2$$
$$A_1 = a_{11}\lambda_1^2 + \sigma_{11}^{*\circ}, \quad A_3 = \mu_{13}\lambda_1^2 + \sigma_{33}^{*\circ}, \quad A_5 = \lambda_1\lambda_3(a_{13} + \mu_{13})$$
$$A_6 = a_{33}\lambda_3^2 + \sigma_{33}^{*\circ}, \quad A_7 = \mu_{13}\lambda_3^2 + \sigma_{11}^{*\circ}, \quad \varkappa_2^2 = \rho\omega^2\mu_{33}^{-1}$$

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Here $I_{jn} = I_j(\sigma_n)$ $(j = 0, 1, n = 1, 2)$ are the modified Bessel functions, ρ is the density of the medium, a_{ij} and μ_{ij} are the coefficients characterizing the relations connecting the stresses and deformations determined by the elastic potentials /1,2/. Their concrete forms corresponding to the types of elastic potential considered in the paper, will be given below.

The contour Γ lies on the real axis and deviates from it only when going round the negative singularities of the function $K_{jn}(u)$ (1.3) from above, and round the positive singularities from below. The choice of the contour is governed by the principle of limit absorption /4/.

The functions $K_{jn}(u)$ (1.3), (1.4) are meromorphic in the complex plane, real on the real axis, and have on it a finite number of zeros and poles, the number depending on $\varkappa_2$ ($K_{nn}(u)$ are even and $K_{jn}(u)$ $(j \neq n)$ are odd). The following expression holds when $u \to \infty$:

$$K_{jn}(u) = c_{jn}u^{-1} + d_{jn}u^{-2} + O(u^{-3}) \qquad (1.6)$$

The constants appearing in (1.6) depend on the cylinder material characteristics and on the magnitude of the initial stress. Their actual form can be obtained simply using the expansions of the Bessel functions, but they are cumbersome and therefore omitted.

2. The relations (1.2)—(1.4) make possible the study of the effect of the initial deformation intensity on the wave process within the cylinder. As we know /4,5/, the behavior of the free surface can be described with sufficient accuracy by means of the formulas

$$\varphi^{\pm}(x) = \varphi(\pm x - a), \quad \pm x - a \gg 1 \qquad (2.1)$$

$$\varphi(t) = \sum_{k=1}^{n} R_k e^{-iz_k t} + O(e^{-Bt}) \qquad (2.2)$$

where z_{k0} and $z_{k\sigma}$ are roots of the equation $\Delta(u, \varkappa_2) = 0$ at $\sigma_{33}{}^{*\circ} = 0$ and $\sigma_{33}{}^{*\circ} = S$, respectively, and R_k are numerical factors.

It is clear that the poles of the functions $K_{jn}(u)$ (1.3) are connected with the phase velocities of the waves propagating along the cylinder surface by the relations $V_k = \omega/z_k$. This enables us to estimate the effect of the initial stresses on the phase velocities of the surface waves, by analysing the effect of the initial deformation intensity on the distribution of zeros of the function $\Delta(u, \varkappa_2)$ (1.4).

Let us assume that the initial state is defined by the condition

$$\sigma_{11}{}^{*\circ} = \sigma_{22}{}^{*\circ} = 0, \quad \sigma_{33}{}^{*\circ} = S = \text{const.}$$

The constants a_{ik} and μ_{ik} appearing in the expressions for the integral operators will have the form

$$\begin{aligned}
&a_{ii} = \lambda + 2\mu + k_0 S a_{ii}{}^\circ, \quad i = 1, 3 \\
&a_{1i} = \lambda + k_0 S a_{1i}{}^\circ, \quad \mu_{1i} = \mu + k_0 S m_{1i}{}^\circ, \quad i = 2, 3 \\
&a_{11}{}^\circ = 2a + 2(1-\gamma)b - \gamma c, \quad a_{33}{}^\circ = 2a + (6 + 4\gamma)b + 2(1+\gamma)c \\
&a_{12}{}^\circ = a - \gamma b, \quad a_{13}{}^\circ = 2a + 2(3 + 2\gamma)b + 2(1+\gamma)c \\
&m_{12}{}^\circ = b - \gamma c/2, \quad m_{13}{}^\circ = b + (2+\gamma)c/4 \\
&\lambda_1{}^2 = 1 - k_0\gamma S, \quad \lambda_3{}^2 = 1 + 2k_0(1+\gamma)S \\
&\gamma = \lambda/\mu, \quad k_0 = (3\lambda + 2\mu)^{-1}, \quad \mu_{33} = \mu
\end{aligned}$$

in the case of the Murnaghan potential (with only the linear terms retained /2/) where λ and μ are the Lamé coefficients and a, b, c are third order constants appearing in the expression for the Murnaghan potential, and the form

$$\begin{aligned}
&a_{ii} = \lambda\Lambda_i{}^{-2} + (2\mu - \lambda k_0 S\Delta^{-1})\Lambda_i{}^{-3}, \quad i = 1, 3 \\
&a_{13} = \lambda/(\Lambda_1\Lambda_3), \quad \mu_{13} = (2\mu - \lambda k_0 S\Delta^{-1})/[\Lambda_1\Lambda_3(\Lambda_1 + \Lambda_3)] \\
&\Lambda_1 = 1 - 0{,}5\, k_0\gamma S\Delta^{-1}, \quad \Lambda_3 = 1 + (1+\gamma)k_0 S\Delta^{-1} \\
&\Delta = 1 - (1+\gamma)Sk_0
\end{aligned}$$

in the case of a harmonic type potential /1/.

Figures 1 and 2 depict the plots of $\eta = (z_{k0} - z_{k\sigma})\cdot 10^3/z_{k\sigma}$ for the steel 09G2S and alloy AMG-6, respectively. The solid lines refer to the Murnaghan potential and the dashed lines to the harmonic potential. Numbers 1, 2 and 3 denote the curves for the values $S = 5\cdot 10^{-4}\mu, 10^{-3}\mu, 5\cdot 10^{-3}\mu$. We see that in the case of steel an increase in the initial stress leads to increased phase velocity exceeding its dependence on the form of the elastic potential. For the alloy this is not true, as the change in the form of the elastic potential alters completely the character of the dependence of the phase velocity of the wave on the initial stress. The latter fact shows that when the wave processes in the initially deformed bodies are investigated, then not

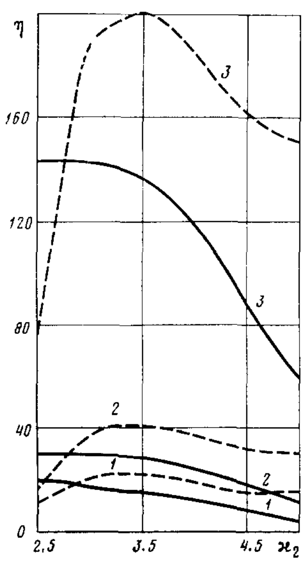
Fig.1

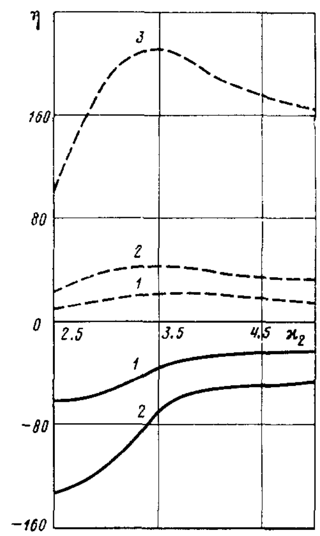
Fig.2

only the initial deformation intensity is of great importance, but also the choice of the form of the elastic potential. The choice should be justified for each particular case.

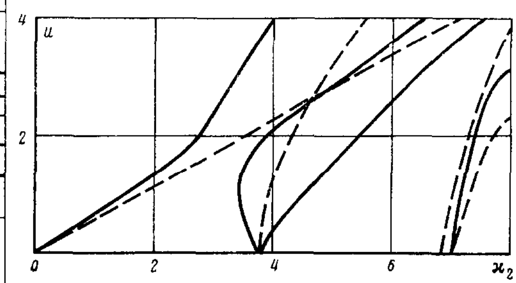
Fig.3

3. We shall investigate the effect of initial stresses on the wave process in the cylinder in more detail, using the problem of radial vibration of the strap. The solution of this problem can be reduced to solving the integral equation

$$\int_{-a}^{a} q(\xi) k_{11}(z-\xi)\, d\xi = 2\pi\mu U_r(z), \quad |z| \leqslant a \tag{3.1}$$

As before, $q(\xi)$ are the contact stresses, $U_r(z)$ denote the displacements of the cylinder surface under the strap, and $k_{11}(t)$ is given by the formula (1.2)—(1.4).

Figure 3 depicts the curves showning the distribution of real zeros (dashed lines) and poles (solid lines) of the function $K_{11}(u)$ for a medium with a Murnaghan potential, for the following values: $\lambda = 926\cdot 10^8$ N/m^2, $\mu = 475\cdot 10^8$ N/m^2, $a = 319\cdot 10^9$ N/m^2, $b = -303\cdot 10^9$ N/m^2, $c = -78{,}4\cdot 10^9$ N/m^2 and $S = 0$. Numerical analysis shows that when $S \neq 0$ the qualitative aspect of the graph remain the same, and that implies /4,5/ that (3.1) has a unique solution in L_α, $\alpha > 1$.

4. Knowing the distribution of zeros and poles of $K_{11}(u)$, we can construct on approximate solution of the integral equation (3.1). Let us replace $K_{11}(u)$ by the function /4,5/

$$K^*(u) = c_{11}(u^2 + B^2)^{-1/2} \prod_{k=1}^{n} (u^2 - z_k^2)(u^2 - \zeta_k^2)^{-1} \tag{4.1}$$

Here $B \gg 1$ is a given approximation parameter, ζ_k $(k = 1, 2, \ldots, m)$ and z_k $(k = 1, 2, \ldots, m_1)$ are real poles and zeros and $K_{11}(u)$, ζ_k $(k = m+1, \ldots, n)$ z_k $(k = m_1 + 1, \ldots, n)$ are complex numbers which can be obtained from the condition of least deviation of $K^*(u)$ from $K_{11}(u)$ on the real axis /4,5/. In (2.1) and (2.2) we gave a schematic form of the solution for the case $U_r(x) = \exp(i\eta x)$ and of the approximating function (4.1). The contact stresses can be written in the form (N_k are numerical coefficients)

$$q(x) = K_{11}^{-1}(0) + \sum_{k=1}^{n} N_k [\exp(iz_k(a+x)) + \exp(iz_k(a-x))] + O[\exp(-B(a-|x|))]$$

The solutions in more detailed form as well as the formulas for N_k and R_k can be found in /4,5/.

Figure 4 gives the computer-derived graphs of the functions $\mathrm{Re}\, q_0 = \mathrm{Re}\, q_0\mu^{-1}$ for $\eta = 0$ (plane strap case) and $a = 7$, $\varkappa_2 = 5.5$ $S = 0$ (a dot-dash line). The curves *1*, *2*, and *3* correspond to the quantity $\zeta = (\mathrm{Re}\, q_\sigma - \mathrm{Re}\, q_0)\cdot 10^2$ for $S = 5\cdot 10^{-4}\mu$, $10^{-3}\mu$, $5\cdot 10^{-3}\mu$.

Figure 5 depicts the graphs illustrating the displacement of the free surface of the cylinder. The dot-dash line corresponds to $\varphi_0(t)$ (2.2) with $S = 0$, line 1 to $(\varphi_\sigma - \varphi_0)\cdot 10^2$ with $S = 5\cdot 10^{-3}\mu$ and line 2 to 1, $(\varphi_\sigma - \varphi_0)\cdot 10^2$ with $S = 10^{-3}\mu$. We see that when the initial stress increases, so does sharply its influence, and the greater the absolute value of the derivative of $\varphi(t)$, the greater the influence.

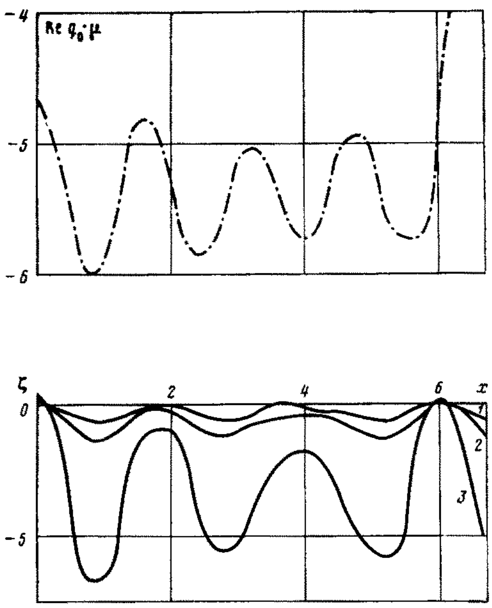

Fig.4

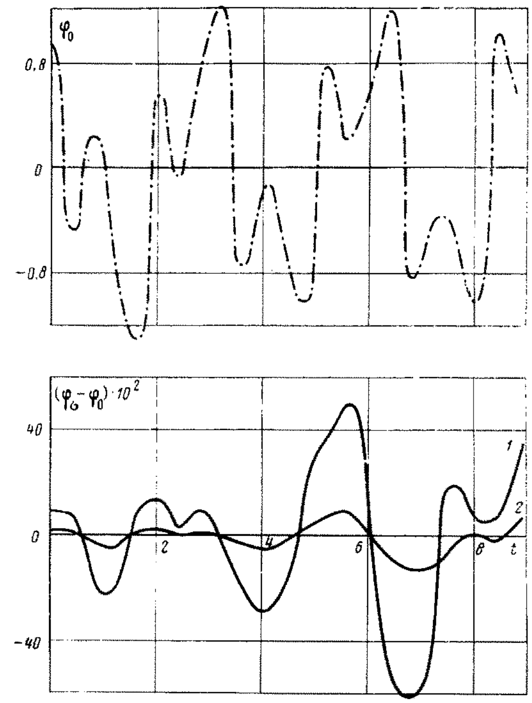

Fig.5

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